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On Some Properties of Middle Cube Graphs and Their Spectra

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Abstract:

In this research paper we begin with study of family of n dimensional hyper cube graphs and establish some properties related to their distance, spectra, and multiplicities and associated Eigen vectors and extend to bipartite double graphs [11].In a more involved way since no complete characterization was available with experiential results in several inter connection networks on this spectra our work will add an element to existing theory.

Keywords: Middle cube graphs, distance-regular graph, antipodal graph, bipartite double graph, extended bipartite double graph, Eigen values, Spectrum, Adjacency Matrix.3

Introduction:

An n dimensional hyper cube Q_n [24] also called n-cube is an n dimensional analogue of Square and a Cube. It is closed compact convex figure whose 1-skelton consists of groups of opposite parallel line segments aligned in each of spaces dimensions, perpendicular to each other and of same length.

1.1) A point is a hypercube of dimension zero. If one moves this point one unit length, it will sweep out a line segment, which is the measure polytypic of dimension one. If one moves this line segment its length in a perpendicular direction from itself; it sweeps out a two-dimensional square. If one moves the square one unit length in the direction perpendicular to the plane it lies on, it will generate a three-dimensional cube. This can be generalized to any number of dimensions. For example, if one moves the cube one unit length into the fourth dimension, it generates a 4-dimensional measure polytopes or tesseract.

The family of hypercube is one of the few regular polytopes that are represented in any number of dimensions. The dual polytopes of a hypercube is called a cross-polytypic.

A hypercube of dimension n has $2n$ "sides" (a 1-dimensional line has 2 end points; a 2-dimensional square has 4 sides or edges; a 3-dimensional cube has 6 faces; a 4-dimensional Thus 8 cells). The number of vertices (points) of a hypercube is 2^n (a cube has 2^3 vertices, for instance).

The number of m-dimensional hyper cubes on the boundary of an n-cube is

$$2^{n-m}\binom{n}{m}$$

For example, the boundary of a 4-cube contains 8 cubes, 24 squares, 32 lines and 16 vertices.

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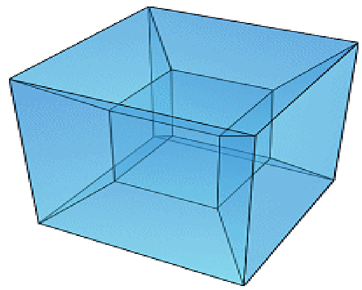

A projection of hypercube into two-dimensional image

A unit hyper cube is a hyper cube whose side has length 1 unit whose corners are

$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$

2^n Points in R^n with each coordinate equal to 0 or 1 termed as measure polytypic.

The correct number of edges of cube of dimension n is $n \times 2^{n-1}$ for example 7-cube has 7×2^6=448 edges.

1.2) Dimension of the cube

	1	2	3	4	5	6
No. of vertices	2	4	8	16	32	64
No. edges	1	4	12	32	80	192

Here we define adjacency matrix of n cube described in a constructive way.

$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$

Since $n\ Q_n$ is n regular bipartite graph of 2^n vertices characteristic vector of subsets of $[n] = \{1, 2, 3, \ldots n\}$ vertices of layer L_k corresponds to subsets of cardinality k.

If n is odd n=2k-1,the middle two layers L_k, L_{k-1} of Q_n with nc_k, nc_{k-1} vertices forms middle cube graph MQ_k by induction.

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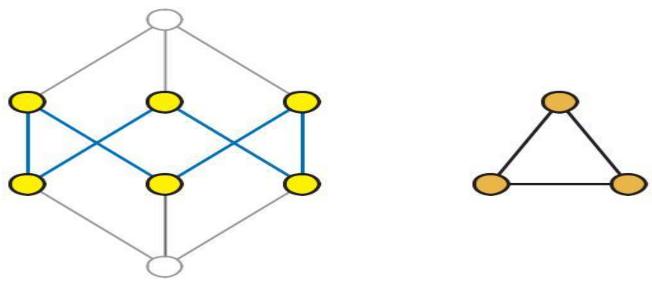
As MQ_k is bipartite double graph which is a sub graph of n-cube Q_n induced by vertices whose binary representations have either k-1 or k no. of 1's is of k-regular as shown in figures below

The middle cube graph MQ_2 is a sub graph of Q_3 or is the bipartite double graph of $Q_2 = K_3$.



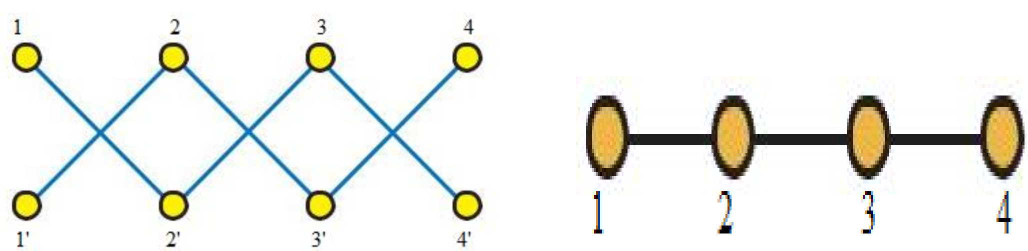
We start with spectral properties of bipartite double graphs [17][18] and extend for study of Eigen values of MQ_k.

1.3) Bipartite double graph: Let H= (V, E) be a graph of order n, with vertex set V = {1, 2…. n}. Its bipartite double graph $1+\lambda, -1-\lambda\hat{H}$ $\bar{H} = (\bar{V}, \bar{E})$ is the graph with the duplicated vertex set

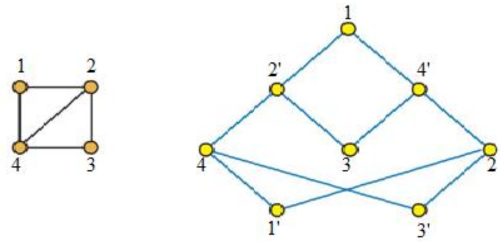
$\bar{V}$ = {1,2… n.1',2',...n' }and adjacencies induced from the adjacencies in H as follows:

$$i : \;\; j \Rightarrow \begin{cases} i \overset{E}{:} \; j' \\ j \overset{E}{:} \; i' \end{cases}$$

Thus, the edge set of $\bar{H}$ is $\bar{E}$ = {ij' | ij ∈ E}. From the definition, it follows that $\bar{H}$ is a bipartite graph [24.21] with stable subsets V_1 = {1, 2…. n}, and V_2 = {1',2',...n' }. For example, if H is a bipartite graph, then its bipartite double graphs $\bar{H}$ consists of two non-connected copies of H.



Path p-4 and its bipartite Double Graph



Graph H has diameter 2 and $\bar{H}$ has diameter 3

If H is a δ-regular graph, then $\bar{H}$ also, if the degree sequence of the original graph H is $\delta = (\delta_1, \delta_2, \delta_3 .... \delta_n)$, the degree sequence for its bipartite double graph is $\bar{\delta} = (\delta_1, \delta_2, \delta_3 .... \delta_n, \delta_1, \delta_2, \delta_3 .... \delta_n)$

The distance between vertices in the bipartite double graph H can be given in terms of even and odd distances in H.

$$dist_{\bar{H}}(\mathrm{i, j}) = \mathrm{dist}^{+}_{H}(\mathrm{i, j})$$

$$dist_{\bar{H}}(\mathrm{i, j'}) = \mathrm{dist}^{-}_{H}(\mathrm{i, j})$$

Involutive auto Orphism without fixed edges, which interchanges vertices i and i' , the map from $\bar{H}$ Onto H defined $i' \rightarrow i, i \rightarrow i$ is a 2-fold covering.

If $\hat{H}$ is extended bipartite double graph by adding edges (i ,i')

for each $i \in V$ $\bar{H} \equiv \hat{H}$.

1.4) Notations:

The order of the graph G is n = {V} and its size is m = {E}. We label the vertices with the integers 1,2,…, n. If i is adjacent to j, that is, ij$\in$E, we write i : j or i $\overset{(E)}{:}$ j. The distance between two vertices is denoted by dist(i,j). We also use the concepts of even distance and odd distance between vertices, denoted by dist+ and dist -, respectively. They are defined as the length of a shortest even (respectively, odd) walk between the corresponding vertices. The set of vertices which are L-apart from vertex i, with respect to the usual distance, is $\Gamma_l(i) = \{ j : dist(\mathrm{i, j}) = l$, so that the degree of vertex is simply $\Gamma_l(i)$. The eccentricity of a vertex is ecc(i)= $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i, j})$ max1j_n dist(i; j) and the diameter of the graph is D =D(G) $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i, j})$ graph G` has the same vertex set as G and two vertices are adjacent in G` if and only if they are at distance l in G. An antipodal graph G is a connected graph of diameter D for which GD is a disjoint union of cliques. The folded graph of G is the graph G whose vertices are the maximal cliques.

Let G = (V;E) be a graph with adjacency matrix A and λ-eigenvector v. Then, the charge of vertex i$\in$ V is the entry vi of v, and the equation $Av = \lambda v$. $\Rightarrow$Eigen values of the bipartite double graph[11,16] $\bar{G}$ and the extended bipartite double graph $\hat{G}$ as functions of the Eigen values of a non-bipartite graph G.

2) Theorem: Let F be a field and let R be a commutative sub ring of F^{n*n} , the set of all n*n Matrices over F. Let M $\in$ R^{m*m} , then

$$\det_F (\mathrm{M}) = \det_F(\det_R(\mathrm{M}))$$

$$\therefore \det_F (\mathrm{M}) = \det_F(\mathrm{AD - BC}).$$

for a bipartite double graph characteristic polynomial. [13]

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We prove the following theorems showing geometric multiplicities of Eigen value λ of H $\Rightarrow$ geometric multiplicities of Eigen values λ and $-\lambda$ of $\bar{H}$
$1+\lambda, -1-\lambda$ of $\hat{H}$

2.1) Theorem: Let H be a graph on n vertices, with the adjacency matrix A and characteristic

$$\bar{u} = u_i^+ v_i (1+\lambda):$$

$$(u^+)_{i'} = \sum_{j:\, i'}^{E} u \overset{+}{j} = \sum_{j:\, i}^{E} v_j = \lambda v_i = \lambda u_i^+$$

Polynomial $\varnothing_H$ (x). Then, the characteristic polynomials of $\bar{H}$ and $\hat{H}$ are, respectively,

$$\varnothing_{\ddot{H}}(x) = (-1)^n \varnothing_H(x)\varnothing_H(-x),$$

$$\varnothing_{\hat{H}}(x) = (-1)^n \varnothing_H(x-1)\varnothing_H(-x-1).$$

Adjacency matrices are, respectively,

$$\ddot{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \text{and } \hat{A} = \begin{pmatrix} O & A+I \\ A+I & O \end{pmatrix}.$$

By above corollary

$$\varnothing_{\ddot{H}}(x) = \det(xI_{2n} - \ddot{A}) = \det\begin{pmatrix} xI_n & -A \\ -A & xI_n \end{pmatrix} = \det(x^2 I_n - A^2)$$

$$= \det(xI_n - A)\det(xI_n + A) = (..1)^n \varnothing_H(x)\varnothing_H(-x);$$

Whereas, the characteristic polynomial of $\hat{H}$ is

$$\varnothing_{\ddot{H}}(x) = \det(xI_{2n} - \hat{A}) = \det\begin{pmatrix} xI_n & -A-I_n \\ -A-I_n & xI_n \end{pmatrix}$$

$$= \det(x^2 I_n - (A+I_n)^2) = \det(xI_n - (A+I_n))\det(xI_n + (A+I_n))$$

$$= \det((x-1)I_n - A)(-1)^n \det(-(x+1)I_n - A)$$

$$= (-1)^n \varnothing_H(x-1)\varnothing_H(-x-1).$$

2.2) Theorem: Let H be a graph and v a λ-eigenvector H. Let us consider the vector u+ with Components $u_i^+ = u_{i'}^+ = v_i$, u- with components $u_i^- = v_i$ and $u_{i'}^- = -v_i$ for $1 \le i, i' \le n$

Then,

$u^+ \lambda$-eigenvector $\bar{H}$ and $(1+\lambda)$ eigenvector $\hat{H}$

$\bar{u}$ - λ-eigenvector $\bar{H}$ and $(-1-\lambda)$ eigenvector $\hat{H}$

Given vertex i, $1 \le i \le n$, all its adjacent vertices are of type j', with i (E) : j.
Then

$$(\mathrm{A}u^{+})_{i} = \sum_{\substack{E \\ j : i'}} u \overset{+}{j} = \sum_{\substack{E \\ j : i}} v_{j} = \lambda v_{i} = \lambda u_{i}^{+}$$

Given vertex I’, $1 \le i \le n$, all its adjacent vertices are of type j, with i (E) : j.
Then

$$(\mathrm{A}u^{+})_{i'} = \sum_{\substack{E \\ j : i'}} u \overset{+}{j} = \sum_{\substack{E \\ j : i'}} v_{j} = \lambda v_{i} = \lambda u_{i}^{+}$$

By a similar reasoning with u^{-}, we obtain

$$(\mathrm{A}u^{-})_{i} = \sum_{\substack{E \\ j : i'}} u \overset{+}{j}' = -\sum_{\substack{E \\ j : i}} v_{j} = -\lambda u_{i-} \text{ and}$$

$$(\mathrm{A}u^{-})_{i'} = \sum_{\substack{E \\ j : i'}} u \overline{j} = \sum_{\substack{E \\ j : i}} v_{j} = -\lambda u_{i}^{'}$$

$$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$$
$$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$$
$$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$$

$\therefore u^{-}$ Is -λ-eigenvector of bipartite double graph $\overline{H}$.

Also 1+λ,-1-λ are Eigen values for u^{+}, u^{-} Eigen vectors of $\hat{H}$
From the above figures realizing an isomorphism [8, 2] defined by

$$f : V[\tilde{O}_k] \to \mathrm{V}[\mathrm{MQ}_k]$$
$$\mathrm{u} \mapsto \mathrm{u}$$
$$\mathrm{u}' \mapsto \bar{\mathrm{u}}$$

Is clearly directive, according to the definition of bipartite double graph, if u and v' are two vertices of $\tilde{O}_k$.

The middle cube graph $[\mathrm{MQ}_k]$ with D=2k-1 by above corollary is isomorphic to $\tilde{O}_k$.

We prove spectrum of Q_{2k-1} contains all Eigen values of $[\mathrm{MQ}_k]$,

θ_i^{+} = $(-1)^{i}$(k-i) and = $\theta_i^{-} = -\theta_i^{+}$ for $0 \le i \le k-1$

With multiplicities $m(\theta_i^{+}) = \mathrm{m}(\theta_i^{-}) = \dfrac{k-1}{k}\dbinom{2k}{i}$

3) Conclusion:

In Verification of the above results,

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$$spMQ_3 = \{\pm 2, \pm 1^2\}$$
$$spMQ_5 = \{\pm 3, \pm 2^4, \pm 1^5\}$$
$$spMQ_7 = \{\pm 4, \pm 3^6, \pm 2^{14}, \pm 1^{14}\}$$
$$spMQ_9 = \{\pm 5, \pm 4^8, \pm 3^{27}, \pm 2^{48}, \pm 1^{42}\}$$

For highest degree Distance polynomials of $[\mathrm{MQ}_k]$

p5 (3) = p5 (1) = p5 (-1) = 1 and p5 (2) =p5 (-1) = p5 (-3) = -1. Then,

$$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$$
$$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$$
$$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$$

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