MARSHALL-OLKIN EXTENDED EXPONENTIAL FAMILY OF DISTRIBUTIONS

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ABSTRACT

A three-parameter family of exponential distributions is introduced and investigated as an alternative to the two-parameter extensions of exponential distribution,. The Marshall-Olkin exponential distribution is present in this family which is introduced in Marshall and Olkin (1997). As an alternative to exponential distribution, this family can be used. This class of distribution's reliability properties is studied. There are many bio-medical applications are noted with these generalized exponential family of distributions. The odd generalized exponential family and its application is illustrated by means of two real lifetime data sets by Tahir, et.al.(2015). A New Lifetime Exponential-*X* Family of Distributions with Applications to Reliability Data explained by Xiaoyan et.al.,(2020). Also a Generalization of the Exponential Distributions detailed by García (2020). Hence there are many real life application based on these family of distribution in Bio-medical, engineering and insurance research. A Weibull distribution of four parameter is introduced and studied. The Weibull distribution is found in this family, which has a number of appealing attributes. A study of this family of distribution's reliability properties is done. Semi-Weibull family of distributions are introduced and studied as a generalization of the four parameter Weibull family. In order to model the data that exhibit periodic movements, the generalized semi-Weibull the semi-Weibull family is helpful. Within the context of time series, the implementation of these distributions is deliberated. The process, first-order regressive Marshall- Olkin minification is thereby deduced. The discussed methods prove to be applicable to generate autoregressive models with any marginal Marshall-Olkin distribution. The study specifically concentrates on the exponential and Weibull families' time series properties.

Key Words: Exponential distribution, Hazard rate, Reliability, Time series, Weibull distribution.

I. INTRODUCTION

In the analysis of survival data or lifetime, a central role is played by the exponential distributions due to their convenient statistical theory, their crucial lack of memory property and their constant hazard rate. There are observed instances where the one-paramete[1]r family of exponential distribution was insufficient to represent the lifetime data. Gamma, Weibull, Gumble are few distributions among the many that are commonly

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used. A general family of distribution shall be introduced here, where exponential and Marshall-Olkin exponential distributions are included. An introduction and study of a family of distributions encompassing both Weibull and Marshall-Olkin is carried out.
To expand families of distributions, a method of adding parameters to distributions was introduced by Marshall and Olkin (1997). This method of adding parameters to a system of distributions is highly effective. An explanation about Mashall-Olkin Discrete Uniform Distribution was done by Sandhya and Prasanth (2014). By adding one more parameter to this method, a generalization was inferred by Jayakumar and T. Mathew (2008) and applied it to Burr type XII distribution. A Generalized Discrete Unifom Distribution's detailing was done by Sandhya &Prasanth (2016). They also illustrated some situations where the method will be instrumental. The term Marshall-Olkin will be referred to as M-O method/ scheme throughout this paper. [2]

The M-O scheme is as follows: Starting with a survival function $\bar{F}$, the one-parameter family of survival functions $\bar{G}_\alpha(x) = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right] \quad -\infty < x < \infty, 0 < \alpha < \infty.$

$\bar{G}_\alpha$ is called the M-O distribution generated from $\bar{F}$. Marshall and Olkin (1997) have applied this to exponential Weibull case. Jayakumar and T. Mathew (2008) generalized this method and the two-parameter family of survival function $\bar{G}_{\alpha,\gamma}$ is proposed as follows:

$$\bar{G}_{\alpha,\gamma}(x) = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^\gamma \quad -\infty < x < \infty, 0 < \alpha < \infty, 0 < \gamma < \infty. \qquad (1.1)$$

When $\alpha = 1$ we get

$\bar{G}_{1,\gamma}(x) = \left[\bar{F}(x)\right]^\gamma$ and in particular when $\alpha = \gamma = 1$,

We get

$\bar{G}_{1,1}(x) = \bar{F}(x)$.

The probability density function (p.d.f.) is

$$g_{\alpha,\gamma}(x) = \gamma\left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma-1} \frac{\alpha f(x)}{\left[1-\bar{\alpha}\bar{F}(x)\right]^2}$$

The hazard rate function is

$$r_{\alpha,\gamma}(x) = \frac{g_{\alpha,\gamma}(x)}{\bar{G}_{\alpha,\gamma}(x)} = \frac{\gamma f(x)}{\bar{F}(x)\left[1-\bar{\alpha}\bar{F}(x)\right]}. \qquad (1.2)$$

In the second section, we attempted to study some of the three-parameter distribution properties generated by applying the method described in (1.1). A significant characteristic is exhibited by the three-parameter exponential family. The exploration of these properties is based on the reliability perspective. In communication engineering, reliability studies and the like, there is a frequent occurrence of the use of Weibull distribution. The broad application of the Weibull distribution facilitates the advanced study of the distribution theory. The subject matter of the third section is a four-parameter Weibull distribution using our method. In this section, the various properties of this distribution within the reliability practitioner's perspective are studied. In section 4, the study of this distribution's interesting characteristics from the reliability practitioner's point of view and the extension to semi-Weibull distribution is done. In modelling the data that exhibits periodic movements, the generalized semi-Weibull law found to be helpful.

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Another area that has garnered a lot of attention in recent years is autoregressive time series modelling with non-Gaussian marginals. This is because many real data we encounter in practice are non-Gaussian and distorted in nature.We create autoregressive models with M-O Scheme-generated distributions. This could be used to create autoregressive processes with any given F as the marginal. The particular cases of MO exponential and MO Weibull autoregressive processes are also discussed. In Section5, these findings are summarized.

2. A THREE PARAMETER EXPONENTIAL FAMILY

Now consider the exponential family generated by (1.1). That is, in (1.1)when $\bar{F}(x) = e^{-\lambda x}$,

we get $\bar{G}_{\alpha,\gamma}(x) = \left[\frac{\alpha e^{-\lambda x}}{1-\bar{\alpha}e^{-\lambda x}}\right]^{\gamma} = \left[\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right]^{\gamma}$ $,-\infty < x < \infty, 0 < \alpha < \infty, 0 < \gamma < \infty$. The family of distributions with survival function $\bar{G}_{\alpha,\gamma}$ will be referred to as the three-parameter exponential family in the sequel. The density of the tree parameter exponential family is

$$g_{\alpha,\gamma}(x) = \left[\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right]^{\gamma-1} \frac{\gamma\alpha\lambda e^{\lambda x}}{\left[e^{\lambda x}-\bar{\alpha}\right]^{2}} \quad ,-\infty < x < \infty, 0 < \alpha < \infty, 0 < \gamma < \infty .$$

Direct evaluation showed that

Mode(X) = $\frac{1}{\lambda}\ln\left(\frac{\alpha-1}{\gamma}\right)$ if $\alpha > 1$ and 0 otherwise. .

Also, Median(X) = $\frac{1}{\lambda}\ln\left(1+\alpha 2^{1/\gamma}-\alpha\right)$. The r^{th}moment about zero is

$$E\left(X^r\right) = \int_0^\infty x^r \left(\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right)^{\gamma} \frac{\gamma\lambda}{e^{\lambda x}-\bar{\alpha}} e^{\lambda x} dx = \int_0^\infty x^{r-1}\left(\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right)^{\gamma} dx$$

Using power series expansion, we get $\alpha^{\gamma}\frac{\Gamma(r)}{\lambda^r}\sum_{j=0}^{\infty}(\bar{\alpha})^j\binom{-\gamma}{j}(\gamma+j)^{-r}$.

The Tables given below gives the values of E(X) and V(X) for various values of α, γ, for $\lambda = 1$.

Table 2.1 E (X) for various values of α and γ

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	3.768	1.515	0.852	0.559	0.402	0.308	0.246	0.204	0.173	0.149
0.4	4.27	1.898	1.156	0.808	0.611	0.486	0.402	0.341	0.295	0.259
0.6	4.585	2.151	1.367	0.987	0.766	0.623	0.523	0.45	0.394	0.351
0.8	4.816	2.344	1.531	1.13	0.893	0.736	0.626	0.543	0.48	0.43
1	5	2.5	1.667	1.25	1	0.833	0.714	0.625	0.556	0.5
1.2	5.153	2.632	1.783	1.354	1.094	0.919	0.793	0.698	0.624	0.564
1.4	5.285	2.747	1.885	1.446	1.178	0.996	0.864	0.764	0.685	0.622
1.6	5.4	2.848	1.976	1.528	1.253	1.066	0.929	0.825	0.743	0.676
1.8	5.502	2.94	2.058	1.604	1.323	1.13	0.989	0.881	0.795	0.726
2	5.595	3.022	2.133	1.672	1.386	1.189	1.045	0.934	0.845	0.773

From Table 2.1, we can observe that for α = 1 and γ = 1, E (X) = 1, as γ increases E (X) decreases and as α increases E (X) increases. The same can be observed in the case of

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V (X) also Table 2.2. This phenomenon may be very useful in obtaining a variety of properties for the distribution for different values of α and γ, which can be seen later in this article.

Table 2.2 V (X) for various values of α and γ

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	22.24	4.5	1.59	0.72	0.38	0.22	0.14	0.09	0.06	0.04
0.4	23.55	5.27	2.07	1.04	0.6	0.38	0.25	0.18	0.13	0.1
0.6	24.23	5.71	2.38	1.26	0.76	0.5	0.35	0.26	0.2	0.15
0.8	24.68	6.02	2.6	1.43	0.89	0.61	0.44	0.33	0.26	0.2
1	25	6.25	2.78	1.56	1	0.69	0.51	0.39	0.31	0.25
1.2	25.25	6.43	2.92	1.68	1.09	0.77	0.58	0.45	0.36	0.29
1.4	25.44	6.58	3.04	1.77	1.17	0.84	0.63	0.5	0.4	0.33
1.6	25.6	6.71	3.14	1.86	1.25	0.9	0.69	0.55	0.44	0.37
1.8	25.74	6.82	3.23	1.93	1.31	0.96	0.74	0.59	0.48	0.4
2	25.86	6.91	3.31	2	1.37	1.01	0.78	0.63	0.52	0.44

Table 2.3 and 2.4 gives the measure of skewness and kurtosis based on moments of the distribution for various values of α, γ for λ = 1respectively. Here we can observe that as α increases measure of skewness decreases, as γ increases measure of skewness increases but when α and γ both increases measure of skewness decreases. For α = 1, it exhibits all the characteristics of an exponential distribution.

Table 2.1 Measure of skewness β_1 obtained for various values of α and γ

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	5.203	7.308	9.588	11.74	13.57	14.98	15.94	16.5	16.7	16.63
0.4	4.604	5.591	6.544	7.347	7.969	8.418	8.715	8.886	8.959	8.958
0.6	4.313	4.805	5.252	5.607	5.868	6.051	6.169	6.238	6.27	6.275
0.8	4.129	4.329	4.502	4.634	4.729	4.793	4.834	4.859	4.87	4.873
1	4	4	4	4	4	4	4	4	4	4
1.2	3.902	3.755	3.635	3.548	3.488	3.449	3.424	3.409	3.401	3.399
1.4	3.825	3.564	3.355	3.207	3.107	3.041	3	2.975	2.962	2.957
1.6	3.761	3.409	3.131	2.938	2.809	2.726	2.673	2.641	2.624	2.617
1.8	3.708	3.281	2.948	2.72	2.57	2.473	2.412	2.375	2.355	2.347
2	3.663	3.171	2.794	2.539	2.373	2.266	2.199	2.159	2.136	2.127

Table 2.2 Measure of Kurtosis β_2 obtained for various values of α and γ

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	10.46	13.38	16.83	20.37	23.67	26.52	28.78	30.42	31.46	31.99
0.4	9.722	11.08	12.54	13.89	15.03	15.95	16.64	17.12	17.44	17.61
0.6	9.37	10.05	10.73	11.34	11.82	12.2	12.48	12.67	12.8	12.87
0.8	9.152	9.425	9.693	9.919	10.1	10.23	10.33	10.4	10.44	10.47
1	9	9	9	9	9	9	9	9	9	9
1.2	8.886	8.687	8.499	8.349	8.235	8.151	8.091	8.05	8.022	8.004
1.4	8.797	8.444	8.118	7.859	7.667	7.527	7.427	7.358	7.311	7.281
1.6	8.725	8.249	7.815	7.476	7.226	7.046	6.918	6.829	6.769	6.73
1.8	8.665	8.088	7.568	7.167	6.873	6.663	6.515	6.412	6.341	6.295
2	8.614	7.952	7.362	6.911	6.584	6.351	6.187	6.073	5.995	5.943

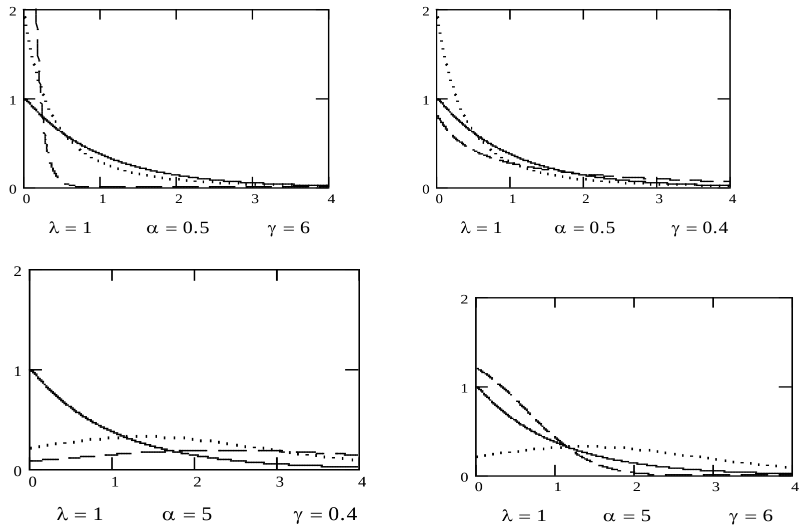

Figure 2.1 The distribution function and density function of the general exponential family of distributions for various values of α and γ

Figure 2.1 given above shows a comparative study of the exponential, Marshall-Olkin exponential and our generalized exponential distributions. The solid lines represents the usual exponential distribution with λ = 1, dotted line represent the Marshall-Olkin family of exponential distribution and the dashed lines represents our general family of exponential distributions. From the figure, it can be seen that for fixed λ and γ, as α increases, the distribution becomes heavy tailed. Also, for fixed λ and α, as γ decreases, the distribution becomes heavy tailed. The deviation from exponential can be seen in the graphs. The Figures show how the distribution can be made more flexible by adjusting the parameters. This adds to the class's richness and makes it better suited to evaluating the various kinds of data sets that are commonly encountered in reliability studies.The hazard rate function is $r_{\alpha,\gamma}(x) = \dfrac{\gamma\lambda e^{\lambda x}}{e^{\lambda x} - \bar{\alpha}}$. The hazard rate exhibits both increasing and decreasing behavior. It can be seen that $r_{\alpha,\gamma}(x)$ is increasing for $\alpha \geq 1$ and is decreasing for $\alpha < 1$.In the case of the Marshall-Olkin exponential, this is also obvious. However, in the Marshall-Olkin exponential case, the rate of increase/decrease in hazard rate differs significantly from that of our generalized exponential family. Figure 2.2 illustrates this.

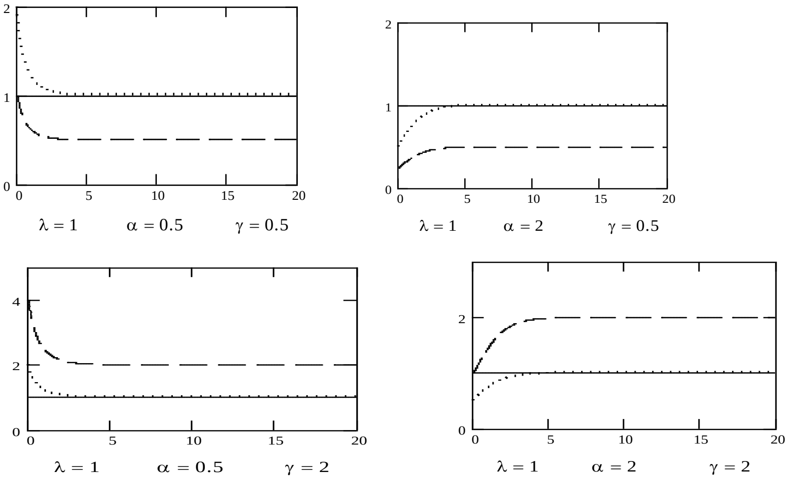

Figure 2.2 Hazard rate of the general exponential family of distributions for various values of α and γ with λ = 1

Figure 2.2 shows a comparison of the hazard rate functions of the exponential, Marshall-Olkin exponential, and the new generalized exponential distribution scheme with $\lambda = 1$.The exponential distribution is represented by solid lines, dotted lines for Marshall-Olkin exponentials, and dashed lines for the general exponential distribution.We can see that the general exponential distribution has a constant hazard rate in the long run, which is a characteristic that has been observed in a variety of real-life situations.

Since the third century A.D., the mean residual life (MRL) has been used. However, in the last two decades, reliability experts, statisticians, and others have exhibitedan increased interestin the MRL. They have produced numerous results related to it. When a unit is of age is represented as t, after time t, the remaining life becomes random. MRL at time t is the predicted value of this random residual life. MRL's huge range of applications is one of its most intriguing features.MRL is used by life insurance actuaries to determine rates and benefits. MRL is used to explore survivorship studies in the biomedical setting. In the social sciences, increasing MRL distributions have been found to be useful as models for the lengths of wars and strikes (see Guess and Proschan, 1988). For any distribution F, the MRL is represented as

$MRL(t) = \frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)dx$ .In our situation this turns out to be $\left(\frac{e^{\lambda t}-\bar{\alpha}}{\alpha}\right)^{\gamma}\int_t^\infty\left(\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right)^{\gamma}dx$. The integral is convergent but very tedious to workout. Computers can be used to evaluate the integral numerically.

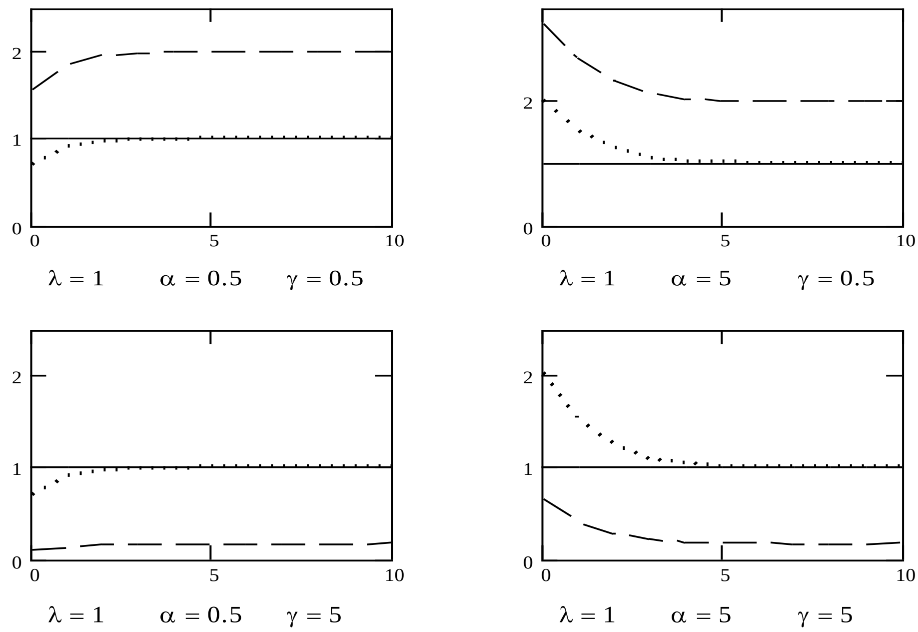

Figure 2.3 A comparative study of the mean residual life of exponential, Marshall-Olkin exponential and the general exponential distributions

Figure 2.3 compares the mean residual lifetime of the exponential, Marshall-Olkin exponential and generalized Marshall-Olkin exponential.The general exponential distributions are represented by a dashed line. The Marshall-Olkin exponential is represented by dotted lines, and the exponential is represented by the solid line. It is to be observed that eventually, the general exponential distribution possesses a constant mean residual lifetime.

The fact that highly uncertain components or systems are intrinsically unrealizable is something that most engineers agree upon. However, they usually lack the ability to quantify uncertainty. For example, it is common among engineers to prepare factors and levels based on this information when designinga system when there is enough information about deterioration and wear of component components.The hazard rate function or MRL function was commonly used to obtain this type of data.However, in order to improve the design, the stability of component parts, along with deterioration, should be considered. The better component, for example, lives longer and has less uncertainty about its residual lifetime.

The basic uncertainty measure for distribution F is differential entropy $H = -\int_0^\infty f(x)\ln\big(f(x)\big)\,dx$. H is commonly referred to as the Shannon's information measure. Intuitively speaking, H gives expected uncertainty contained in f(x) about the predictability an outcome of F. That is, H measures concentration of probabilities. Low entropy distributions are more concentrated, hence more informative that high entropy distributions (see Ebrahimi, 1996). In the case of our general family of distributions, theShannon's measure of entropy is given by

$$H = -\int_0^\infty \left(\left[\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right]^{\gamma+1}\frac{\gamma\lambda e^{\lambda x}}{\alpha}\right)\ln\left(\left[\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right]^{\gamma+1}\frac{\gamma\lambda e^{\lambda x}}{\alpha}\right)dx\,.$$

The table below presents the Shannon's entropy measure of uncertainty of the general exponential family of distribution for various values of α and γ with $\lambda = 0.2$. Note that there is less uncertainty for large values of γ and small values of α. Also, for fixed α, as γ increases, the uncertainty diminishes. Note that for fixed γ, as α increases the uncertainty increases.

Table 2.5 Shannon's measure of entropy for the three-parameter generalized family of exponential distributions for various values of α and γ with λ = 0.2.

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	3.84	2.9	2.33	1.91	1.6	1.34	1.13	0.95	0.8	0.66
0.4	4.03	3.21	2.71	2.36	2.08	1.86	1.67	1.51	1.37	1.24
0.6	4.12	3.36	2.91	2.58	2.33	2.13	1.95	1.8	1.67	1.55
0.8	4.18	3.46	3.03	2.73	2.49	2.3	2.14	2	1.87	1.76
1	4.22	3.53	3.12	2.83	2.61	2.43	2.27	2.14	2.02	1.92
1.2	4.25	3.58	3.19	2.91	2.7	2.52	2.38	2.25	2.14	2.04
1.4	4.27	3.62	3.24	2.97	2.77	2.6	2.46	2.34	2.23	2.13
1.6	4.29	3.65	3.28	3.02	2.83	2.66	2.53	2.41	2.3	2.21
1.8	4.3	3.67	3.32	3.07	2.87	2.72	2.59	2.47	2.37	2.28
2	4.32	3.7	3.35	3.1	2.92	2.76	2.64	2.52	2.43	2.34

A modification was introduced to the Shannon's entropy measure by Ebrahimi (1996).Having information about the component's current age under consideration is common in survival analysis and life tests. In all situations, age must be considered when calculating uncertainty. Shannon's entropy H is obviously unsuitable in such scenarios and must be adjusted to account for the age. In Ebrahimi, a more practical approach is suggested, one that uses the age (1996).

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Given that a component has survived up to time t, the measure of entropy after time t given by $H(t) = 1-\frac{1}{\bar{F}(t)}\int_t^{\infty}\ln\left(\frac{f(x)}{\bar{F}(x)}\right)f(x)dx$. In the case of our exponential family, this is

$$H(t) = 1-\left(\frac{e^{\lambda x}-\bar{\alpha}}{\alpha}\right)^{\gamma}\int_t^{\infty}\ln\left(\frac{\gamma\lambda e^{\lambda x}}{e^{\lambda x}-\bar{\alpha}}\right)\left[\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}\right]^{\gamma+1}\frac{\gamma\lambda e^{\lambda x}}{\alpha}dx.$$

The Figure 2.4 gives an idea about the distribution of modified Shannon's entropy about the values t = .5 and t = 5 with λ = 1. From the Figures we can observe that the modified Shannon's entropy remains constant for large t.

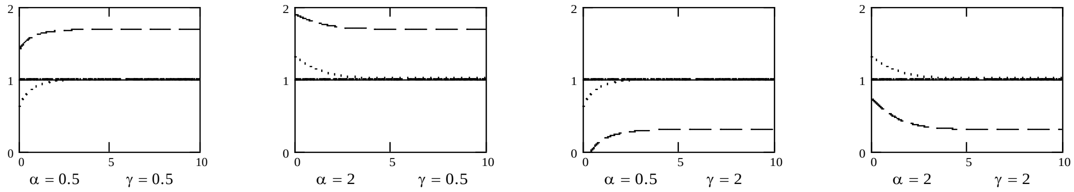

Figure 2.4 Modified Shannon entropy measure for exponential, Marshall-Olkin exponential and the general exponential distributions

The cumulative distribution function G of a non negative random variable is said to be new better than used of specific age t_0 if $\bar{F}(x+t_0)\le\bar{F}(x)\bar{F}(t_0)$ and new worse than used when $\bar{F}(x+t_0)\ge\bar{F}(x)\bar{F}(t_0)$. The general exponential distribution is new worse than used for $\alpha\le 1$ and new better than used if $\alpha\ge 1$.

We can see that the 3 parameter general Marshall-Olkin exponential distribution give a variety of survival characteristics for various values of λ, α and γ but preserves many of the characteristics of exponential distribution in the long run, and therefore will be very useful in reliability analysis, modeling etc….

3. A FOUR-PARAMETER WEIBULL FAMILY

The Weibull distribution was first used to denote the distribution of material breaking strength, and then for a variety of other uses (see Johnson et al.)(1994). The exponential and Rayleigh distributions are special cases of the Weibull distribution.It is well known that the hazard function of this distribution is a decreasing function when the shape parameter β is less than 1, a constant when β = 1 (exponential case), and increasing when β >1. Following Weibull(1951), Kao(1958,1959), and Berrettoni, many writers called for the use of the distribution in reliability and quality control work (1964).
The distribution also becomes suitable when the conditions for strict "randomness" of the exponential distribution are not satisfied, with the shape parameter having a characteristic or predictable value depending on the underlying nature of the problem being considered, due to the nature of the hazard function discussed above.In contrast to the exponential distribution, probabilistic bases for the Weibull distribution are rarely found in contexts where the distribution is used. However, Malik(1975) and Franck(1988) have given the Weibull distribution some simple physical meanings and interpretations, allowing natural applications of this distribution in reliability problems, especially those involving wearing styles. Because the distribution is a power transformation of the exponential, the power β provides a handy way to introduce some flexibility into the model.

In the study of wind speed, the Weibull distribution has been studied by many scholars. The Weibull distribution was used by Pavia and O'Brien (1986) to model wind speed over the ocean, and Carlin and Haslett (1982) used it to model wind power from a scattered arrow of wind turbine generators.Wilks (1989) and Selker and Haith (1990) used the Weibull distribution to model rainfall intensity data, and Wilks (1989) used it to analyze rainfall and flood data.

Many studies in the field of health science have used the Weibull model. Berry(1975), for example, spoke about using the Weibull distribution to design carcinogenesis experiments. Dyer(1975) used the distribution to investigate the relationship between systolic blood pressure, serum cholesterol, and smoking to 14-year mortality in the Chicago Peoples Gas Company; coronary and cardiovascular-renal mortality were also compared in two competing risk models.By taking Doll and Hills data for British Physician, Whittemore and Altschuler(1976) used the model to analyse lung cancer incidence in cigarette smokers.The Weibull model was used to analyse the Stanford heart transplant data by Aitkin, Laird, and Francis (1983). Chen et al.(1985) used the Weibull distribution to perform a Bayesian analysis of survival curves for cancer patients after therapy.

In addition to the problems discussed above, the Weibull distribution has been helpful in several other situations. Fong, Rehm, and Graminski(1977), for example, used the distribution as a microscopic paper degrading model. Ogden suggested the Weibull shelf-life model for pharmacy concerns (1978). In genetic studies, Rinket al.(1979) used the three-parameter Weibull distribution to calculate sweetgun germination data. Ida incorporated the use of the Weibull distribution in the interpretation of reaction time data (1980). Dyer has proved the Weibull distribution's role in offshore oil/gas lease bidding issues (1981).

The Weibull distribution is undeniably the distribution that has received maximum attention during the past decades.

When $\bar{F}(x) = e^{-\theta x^{\beta}}, \quad 0<\theta<\infty, 0<\beta<\infty,$

$$\bar{G}_{\alpha,\gamma}(x) = \left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma} \quad ,0<x<\infty, 0<\alpha<\infty, 0<\beta<\infty, 0<\gamma<\infty, 0<\theta<\infty$$

and
$$g_{\alpha,\gamma}(x) = \left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma} \frac{\gamma\theta\beta e^{\theta x^{\beta}} x^{\beta-1}}{e^{\theta x^{\beta}}-\bar{\alpha}} \quad ,0<x<\infty, 0<\alpha<\infty, 0<\beta<\infty, 0<\gamma<\infty, 0<\theta<\infty.$$

The mode of the distribution is $\left(\frac{1}{\theta}\ln\left(\frac{\alpha-1}{\gamma}\right)\right)^{\frac{1}{\beta}}$ if $\alpha>1$. The median is

$$\frac{\left[\ln\left(\alpha\left(\frac{1}{2}\right)^{-\frac{1}{\gamma}}+\bar{\alpha}\right)\right]^{\frac{1}{\beta}}}{\theta^{\frac{1}{\beta}}}.$$

The r^{th} moment of the distribution about zero is

$$E(X^r) = \int_0^{\infty} x^r \left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma} \frac{\gamma\theta\beta e^{\theta x^{\beta}} x^{\beta-1}}{e^{\theta x^{\beta}}-\bar{\alpha}} dx = \int_0^{\infty} x^{r-1} \left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma} dx$$

Using power series expansion, we get $\frac{\alpha^{\gamma}}{\beta}\frac{\Gamma\left(\frac{r}{\beta}\right)}{\theta^{\frac{r}{\beta}}}\sum_{j=0}^{\infty}(\bar{\alpha})^{j}\binom{-\gamma}{j}(\gamma+j)^{-\frac{r}{\beta}}$.

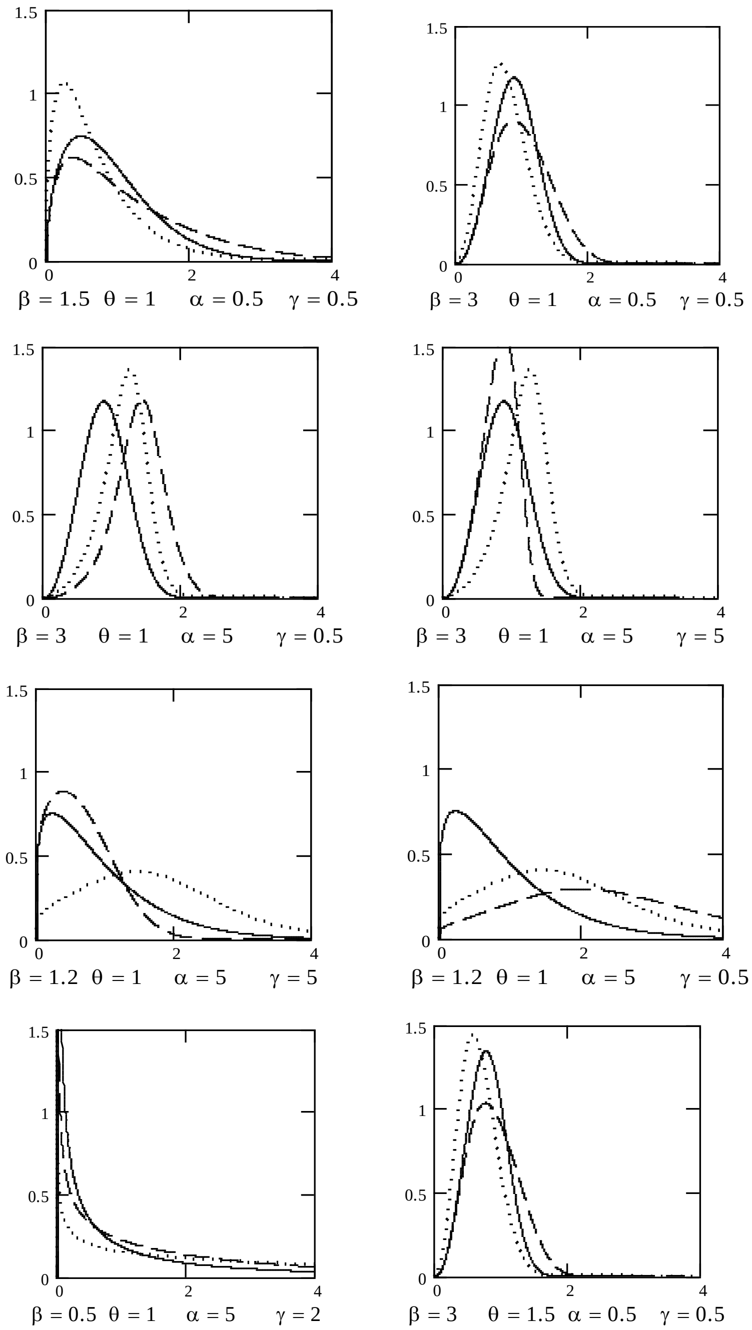

Figure 3.1 A comparative study of the Weibull, Marshall-Olkin Weibull and the general Weibull

A comparative study of the Weibull(sold line), Marshall-Olkin Weibull (doted line) and, and general Weibull (dashed line) is given in Figure 3.1. From the Figures we can observe that for various values of α, γ, β and θ there is very large flexibility in the shape of the probability density function. For γ = 1 we can have the marshal-Olkin Weibull distribution, and for all other values of γ we can have a verity of other shapes for the probability density function.

The tables given below gives the skewness of the general Weibull family of distributions for various values of α, β, θ and γ.

ISSN:0975-3583,0976-2833 VOL12, ISSUE 03,2021

Table 3.1 Measure of skewness of the general Weibull family of distributions

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	0.654	1.192	1.631	1.925	2.085	2.144	2.132	2.075	1.995	1.903
0.4	0.516	0.764	0.949	1.065	1.13	1.157	1.16	1.147	1.125	1.097
0.6	0.455	0.578	0.663	0.716	0.744	0.757	0.76	0.756	0.749	0.739
0.8	0.421	0.47	0.503	0.522	0.532	0.537	0.539	0.538	0.536	0.533
1	0.398	0.398	0.398	0.398	0.398	0.398	0.398	0.398	0.398	0.398
1.2	0.383	0.347	0.325	0.313	0.306	0.303	0.302	0.302	0.303	0.305
1.4	0.371	0.308	0.271	0.251	0.24	0.235	0.233	0.233	0.234	0.236
1.6	0.362	0.277	0.229	0.204	0.19	0.184	0.181	0.181	0.182	0.184
1.8	0.356	0.253	0.197	0.167	0.152	0.144	0.141	0.141	0.142	0.144
2	0.35	0.233	0.17	0.138	0.122	0.114	0.111	0.11	0.111	0.113

The tables given below gives the Kurtosis of the general Weibull family of distributions for θ = 1 and β = 2 with various values of α and γ .

Table 3.2 Measure of kurtosis of the general Weibull family of distributions

α \ γ	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	3.25	3.968	4.698	5.287	5.703	5.96	6.088	6.119	6.084	6.005
0.4	3.206	3.532	3.853	4.104	4.279	4.393	4.458	4.489	4.494	4.482
0.6	3.211	3.369	3.523	3.641	3.724	3.779	3.812	3.831	3.838	3.838
0.8	3.226	3.289	3.349	3.395	3.427	3.449	3.463	3.471	3.475	3.476
1	3.245	3.245	3.245	3.245	3.245	3.245	3.245	3.245	3.245	3.245
1.2	3.264	3.221	3.179	3.147	3.124	3.109	3.099	3.092	3.088	3.086
1.4	3.283	3.207	3.135	3.08	3.041	3.014	2.995	2.983	2.975	2.971
1.6	3.3	3.201	3.105	3.032	2.981	2.945	2.92	2.903	2.892	2.885
1.8	3.317	3.199	3.085	2.999	2.937	2.894	2.863	2.843	2.828	2.819
2	3.333	3.201	3.072	2.974	2.905	2.856	2.821	2.796	2.78	2.768

The hazard rate is $r_{\alpha,\gamma}(x) \quad = \quad \dfrac{\gamma\theta\beta e^{\theta x^{\beta}} x^{\beta-1}}{e^{\theta x^{\beta}} - \bar{\alpha}}$

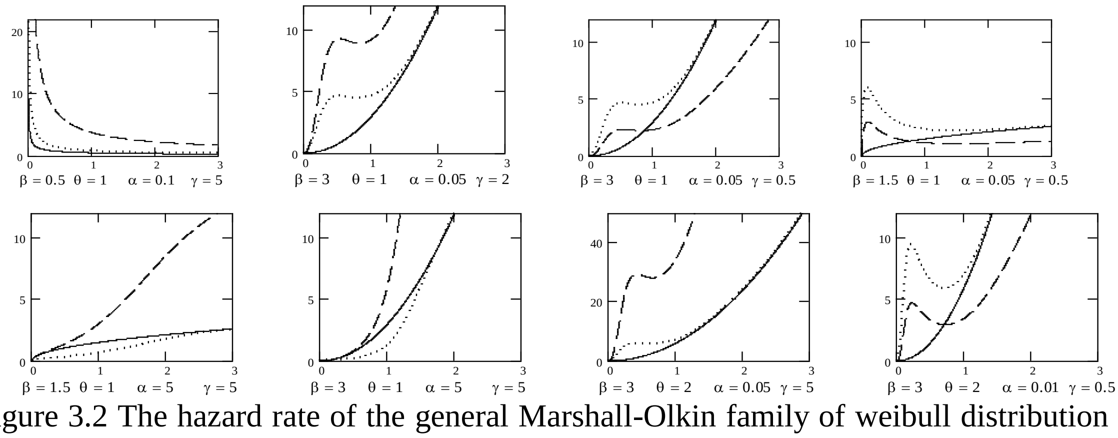

Figure 3.2 The hazard rate of the general Marshall-Olkin family of weibull distribution for various values of α and γ with θ = 1 and β = 2.

In Fig. 3.2, a comparative study of the hazard rate functions of the Weibull, Marshall-Olkin Weibull and the new generalised Weibull is given. The Weibull distribution is represented by solid line, Mashall-Olkin Weibull by dotted lines and Weibull distribution by dashed lines.For various values of α, β, θ and γ the hazard function exhibit different characteristicsThe hazard function decreases for some parameter values, increases for others, and exhibits non-monotone characteristics for still others. Besides, unlike the traditional Weibull law, this one accounts for a wide range of changes in the ageing process.As a result, the distribution has a wide range of reliability characteristics.

ISSN:0975-3583,0976-2833 VOL12, ISSUE 03,2021

The mean residual life (MRL) time is given by the equation

$$\text{MRL}(t) = \frac{1}{\bar{F}(t)}\int_t^{\infty}\bar{F}(x)dx = \left(\frac{e^{\theta x^{\beta}}-\bar{\alpha}}{\alpha}\right)^{\gamma}\int_t^{\infty}\left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma}dx .$$

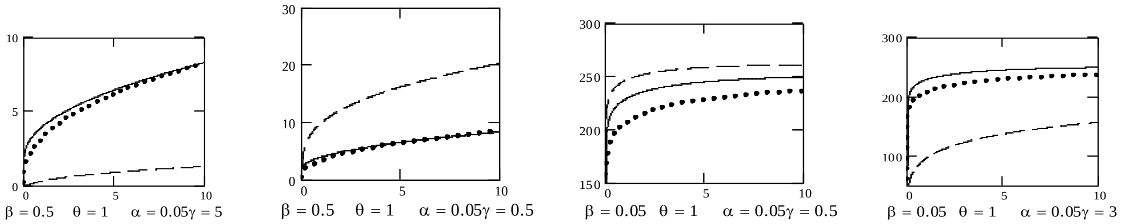

Figure 3.3 Plot of the MRL of Weibull, Marshall-Olkin Weibull and General Weibull

The integral is convergent but very tedious to workout. Numerical evaluation of the integral is possible using computers.

The Shannon's measure of uncertainty is $H = -\int_0^{\infty} f(x)\ln\left(f(x)\right)dx$. That is

$$H = -\int_0^{\infty}\left(\left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma}\frac{\gamma\theta\beta e^{\theta x^{\beta}}x^{\beta-1}}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)\ln\left(\left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma}\frac{\gamma\theta\beta e^{\theta x^{\beta}}x^{\beta-1}}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)dx$$

Event Hugh the expression is convergent It seems to be very tedious to evaluate it. But it is possible to find the values of the integral for various values of α and γ. The table 3.3 present the Shannon's entropy measure of uncertainty of the general exponential family of distribution for various values of α and γ with θ = 1 and β = 2.

Table 3.3 Shannon's entropy measure of uncertainty of the general exponential family of distribution

0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.2	1.361	0.899	0.6	0.382	0.211	0.072	-0.043	-0.141	-0.227	-0.302
0.4	1.394	0.991	0.738	0.554	0.409	0.291	0.192	0.106	0.031	-0.036
0.6	1.401	1.026	0.798	0.632	0.503	0.397	0.307	0.23	0.162	0.101
0.8	1.402	1.044	0.831	0.678	0.559	0.461	0.378	0.307	0.244	0.188
1	1.4	1.054	0.851	0.707	0.595	0.504	0.427	0.36	0.302	0.249
1.2	1.397	1.059	0.864	0.727	0.622	0.536	0.463	0.4	0.344	0.295
1.4	1.394	1.062	0.873	0.742	0.641	0.559	0.49	0.43	0.377	0.33
1.6	1.39	1.063	0.879	0.753	0.656	0.577	0.511	0.454	0.404	0.359
1.8	1.387	1.063	0.884	0.761	0.667	0.592	0.529	0.474	0.425	0.382
2	1.383	1.063	0.887	0.767	0.677	0.604	0.543	0.49	0.443	0.402

The measure of entropy after time t given by Ebrahimi (1996) is

$$H(t) = 1-\frac{1}{\bar{F}(t)}\int_t^{\infty}\ln\left(\frac{f(x)}{\bar{F}(x)}\right)f(x)dx$$

$$H(t) = 1-\left(\frac{e^{\theta t^{\beta}}-\bar{\alpha}}{\alpha}\right)^{\gamma}\int_t^{\infty}\ln\left(\frac{\gamma\theta\beta e^{\theta x^{\beta}}x^{\beta-1}}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{2}\left(\frac{\alpha}{e^{\theta x^{\beta}}-\bar{\alpha}}\right)^{\gamma}dx$$

When data shows non-proportional risks, the odds ratio and proportional odds' use is becoming more common in engineering reliability and biological survival analysis.However, in some cases where survival data indicates a non-monotone hazard rate, either proportional hazard or proportional odds modelling may fall short of accurately describing the situation.Wang et al. (2003) suggest the log odds rate (LOR) to describe the failure distribution, to provide a graphical examination of cases where survival data suggest a non-monotone hazard rate but a monotone log-odds rate, and to propose the log odds rate as a new way of watching and modelling the failure process in the ageing area.
The monotone Log-Odds rate (Yao et al. (2003)) is

$$\mathrm{LOR}(t) \;=\; \frac{f(t)}{F(t)\bar{F}(t)} .= \frac{\gamma\theta\beta e^{\theta t^{\beta}} x^{\beta-1}}{e^{\theta t^{\beta}}-\bar{\alpha}}\left(\frac{e^{\theta t^{\beta}}-\bar{\alpha}}{\alpha}\right)^{\gamma}.$$

The distribution of Log-Odds rate of Weibull, Marshall-Olkin Weibull and the general Weibull distribution for various values of α, γ, θ and β is plotted in Figure 3.4.

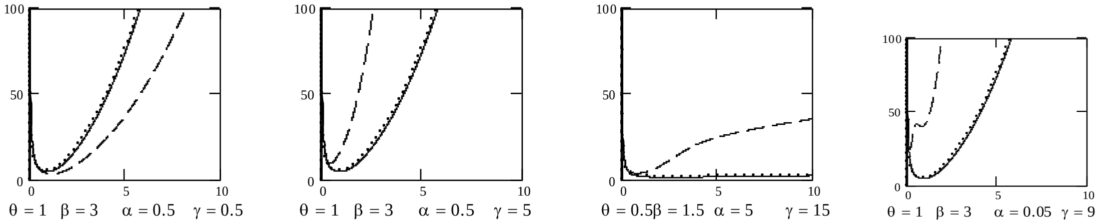

Figure 3.4 Log odds rate distribution of Weibull, Marshall-Olkin Weibull and the general Weibull distribution for various values of α and γ with θ = 1 and β =2.

4. The general semi Weibull distribution

A random variable X with positive support is said to follow semi-Weibull distribution if its survival function is given by $\bar{F}(x) = e^{-\psi(x)}$ where $\psi(x)$ satisfies the functional equation $\psi(x) = x^{\beta}h(x)$ where h(x) is periodic in ln(x) with period $-\frac{2\pi\beta}{\ln(p)}$ (see Jose (1991)). For example, $h(x) = e^{(\nu \mathrm{Cos}(\alpha \ln(x)))}$, $0<\nu<1$ is periodic with period $e^{-2\pi}$ and $\psi(x)$ is monotone increasing. When ν = 0 semi Weibull become Weibull. The general semi Weibull distribution is defined as

$$\bar{G}(x) \;=\; \left(\frac{\alpha}{e^{\psi(x)}-\bar{\alpha}}\right)^{\gamma} \;\&\; \quad g(x) \;=\; \frac{\gamma}{\alpha}\left(\frac{\alpha}{e^{\psi(x)}-\bar{\alpha}}\right)^{\gamma+1} e^{\psi(x)}\psi'(x)$$

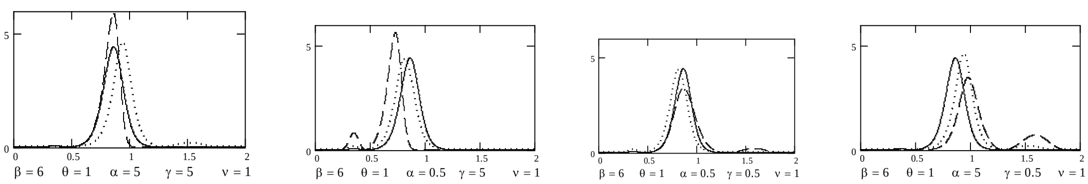

Figure 4.1 Plot of the semi-Weibull, Marshall-Olkin semi-Weibull and general semi-Weibull distribution for various values of parameters

In Figure 4.1.,a comparative study of the plot of the semi-Weibull (solid line), Marshall-Olkin semi-Weibull (dotted line) and general semi-Weibull (dashed line) distribution is given.We can see from the figure that, in addition to the Weibull and Marshall Olkin Weibull distributions, the general Weibull distribution can take on a variety of shapes for

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representing periodic data.The hazard rate for the h(x) given above is

$$r(x) = \frac{\gamma\theta x^{\beta-1}\beta\left(\nu\sin\left(\beta\ln(x)\right)-1\right)e^{\nu\cos\left(\beta\ln(x)\right)+\theta x^{\beta}e^{\nu\cos\left(\beta\ln(x)\right)}}}{e^{\theta x^{\beta}e^{\nu\cos\left(\beta\ln(x)\right)}}-\bar{\alpha}}.$$

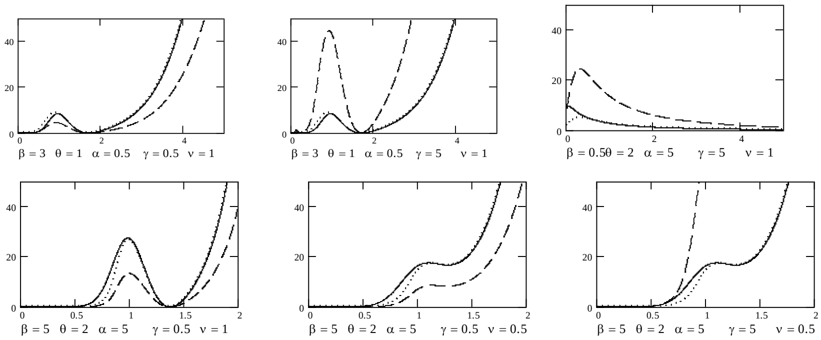


Figure 4.2 Plot of the hazard rate for semi--Olkin semi-Weibull and general semi Weibull distribution

Figure 4.2 represent the plot of the hazard rate function of general semi-Weibull distribution for various values of the parameters. The non-monotone characteristics of the hazard rate indicate immense application in the field of reliability.

5. The Marshall-Olkin first order autoregressive minification process

The study on minification processes began with the work of Tavares (1980). He developed a first order autoregressive exponential minification process. In his work, the observations are generated by the equation

$$X_n = k\min(X_{n-1}, \varepsilon_n), \qquad n \geq 1 \qquad (5.1)$$

where k > 1 is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Because of the structure of (5.1) the process $\{X_n\}$ is called minification process. Sim (1986) developed a first order autoregressive Weibull process and studied its properties. Arnold (1993) developed a logistic process involving Markovian minimization.

Several other minification models have been built so far, with minor changes to (5.1). A first-order autoregressive minification procedure with a Pareto marginal distribution was suggested by Yeh et al. (1988). This was expanded by Pillai (1991) to produce a first-order autoregressive semi-Pareto process. A minification process with logistic marginal distribution was considered by Arnold and Robertson (1989).Such minification processes in general have the structure given by

$$X_n = \begin{cases} kX_{n-1} & \text{w.p.} \quad p \\ k\min(X_{n-1}, \varepsilon_n) & \text{w.p.} \quad 1-p \end{cases}, \quad 0 < p < 1,$$

where 'w.p.' stands for 'with probability'. Pillai, Jose and Jayakumar (1995) introduced another minification process having the form

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ k\min(X_{n-1}, \varepsilon_n) & \text{w.p.} \quad 1-p \end{cases}, \quad 0 < p < 1.$$

Lewis and McKenzie (1991) obtained necessary and sufficient conditions on the hazard rate of the marginal distributions for a minification process to exist.

We describe a first-order autoregressive minification process that can be applied to any distribution with a closed-form survival function. We use it to describe two first-order autoregressive minification processes with Marshall-Olkin exponential and Marshall-Olkin Weibull distributions as marginals and investigate some of their properties. The procedure estimation is also briefed about.

Theorem 5.1. Let $\bar{F}(x)$ be the survival function of a distribution and $\bar{H}(x)$ be the Marshal-Olkin survival function given by

$$\bar{H}(x) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} . \qquad (5.2)$$

Consider the first order autoregressive minification process given by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p. } \alpha \\ \min(X_{n-1}, \varepsilon_n) & \text{w.p. } 1-\alpha \end{cases} . \qquad (5.3)$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$.Then $\{X_n\}$ is stationary Markovian first order autoregressive process with survival function $\bar{H}(x)$ if and only if ε_n has survival function $\bar{F}(x)$.

Proof

From (5.3) it follows that

$$\bar{F}_{X_n}(x) = \alpha\bar{F}_{\varepsilon_n}(x) + (1-\alpha)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x)$$

Under stationary equilibrium

$$\bar{F}_X(x) = \frac{\alpha\bar{F}_{\varepsilon_n}(x)}{1-\bar{\alpha}\bar{F}_{\varepsilon_n}(x)}$$

If we take $\bar{F}_{\varepsilon_n}(x) = \bar{F}(x)$, then it easily follows that

$$\bar{F}_X(x) = \bar{H}(x)$$

which is the Marshall-Olkin survival function.

Conversely if we take $\bar{F}_{X_n}(x) = \bar{H}(x)$, then it is easilyfollows that $\bar{F}_{\varepsilon_n}(x) = \bar{F}(x)$.

Assume that the survival function of X_{n-1} is $\bar{H}(x)$ and the survival function of ε_n is $\bar{F}(x)$, then $\bar{F}_{X_n}(x) = \bar{H}(x)$.

Even if X_0 is arbitrary, it is easy to establish that $\{X_n\}$ is stationary and is asymptotically marginally distributed as $\bar{H}(x)$.

This result can be easily extended to k^{th} order autoregressive case.

Theorem 5.2.

Consider the k^{th} order autoregressive time series model defined by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p. } \alpha_0 \\ \min(X_{n-1}, \varepsilon_n) & \text{w.p. } \alpha_1 \\ \min(X_{n-2}, \varepsilon_n) & \text{w.p. } \alpha_2 \\ . \\ .. \\ . \\ \min(X_{n-k}, \varepsilon_n) & \text{w.p. } \alpha_k \end{cases} \qquad (5.4)$$

where $0 < \alpha_i < 1$, $\alpha_1 + \alpha_2 + ... + \alpha_k = 1 - \alpha_0$. Then $\{X_n\}$ is stationary with survival function $\bar{H}(x)$ if and only if $\{\varepsilon_n\}$ has survival function $\bar{F}(x)$.

Proof

(5.4) in terms of survival functions is

$$\bar{F}_{X_n}(x) = \alpha_0\bar{F}_{\varepsilon_n}(x)+\alpha_1\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x)+...+\alpha_k\bar{F}_{X_{n-k}}(x)\bar{F}_{\varepsilon_n}(x)$$

Under stationary equilibrium

$$\bar{F}_X(x) = \alpha_0\bar{F}_{\varepsilon_n}(x)+\alpha_1\bar{F}_X(x)\bar{F}_{\varepsilon_n}(x)+...+\alpha_k\bar{F}_X(x)\bar{F}_{\varepsilon_n}(x)$$

That is, $\bar{F}_X(x) = \dfrac{\alpha_0\bar{F}_{\varepsilon_n}(x)}{1-\bar{\alpha}_0\bar{F}_{\varepsilon_n}(x)}$.

Theorem is applicable to all types of Marshall-Olkin distributions and therefore we can define the first order autoregressive Marshall-Olkin exponential process.

5.1. First Order Autoregressive Minification Process with Exponential Marginal Distribution

Consider the first order autoregressive minification process given by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad \alpha \\ \min(X_{n-1},\varepsilon_n) & \text{w.p.} \quad 1-\alpha \end{cases}.$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$. Then $\{X_n\}$ is stationary Markovian first order autoregressive Marshall-Olkin exponential process with survival function $\bar{H}(x)$ if and only if ε_n has exponential distribution $\bar{F}(x)$ and $0<\alpha<1$.

$$\begin{aligned} P(X_{n+1}>X_n) &= \alpha P(\varepsilon_{n+1}>X_n)+(1-\alpha)P(\min(X_n,\varepsilon_{n+1})>X_n) \\ &= \alpha P(\varepsilon_{n+1}>X_n) \\ &= \frac{\alpha}{2} \end{aligned}$$

$$\begin{aligned} \bar{F}_{X_n,X_{n+1}}(x,y) &= P\left(X_n>x, X_{n+1}>y\right) \\ &= \left(\alpha\bar{F}_{X_n}(x)+(1-\alpha)\bar{F}_{X_n}(\max(x,y))\right)\bar{F}_{\varepsilon_n}(y) \\ &= \left(\alpha\frac{\alpha}{e^{\lambda x}-\bar{\alpha}}+(1-\alpha)\frac{\alpha}{\max\left(e^{\lambda x},e^{\lambda y}\right)-\bar{\alpha}}\right)e^{-\lambda x}. \end{aligned}$$

$$Cov(X_n,X_{n+1}) = (1-\alpha)V(X_n).$$

$$Corr(X_n,X_{n+1}) = 1-\alpha..$$

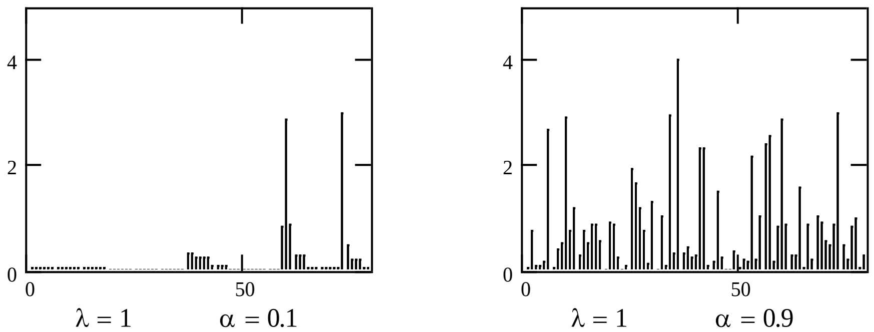


Figure 5.1 Sample path behavior of the first order autoregressive Marshall-Olkin exponential process

Now we describe the estimation of first order autoregressive Marshall-Olkin process. From the observed series find $P(X_{n+1} > X_n)$. If $P(X_{n+1} > X_n)$>.5 then we can conclude that the process is not a good fit. If $P(X_{n+1} > X_n)$ = .5 then α = 1 and we can see hat $Corr(X_n, X_{n+1}) = 1-\alpha = 0$ and $\{X_n\} \underset{=}{d} \{\varepsilon_n\}$

If $P(X_{n+1} > X_n)$<.5 we can estimate the value of α by method of moments. Equate sample correlation to population correlation coefficient 1- α and find the value of α. Then the parameter λ can be estimated using the formula

$$E(X) = -\frac{\alpha \ln(\alpha)}{\lambda \bar{\alpha}}$$

where X is a random variable following Marshall-Olkin exponential distribution (see Marshall and Olkin (1997)).

5.2. First Order Autoregressive Minification Process With Weibull Marginal Distribution

Consider the first order autoregressive minification process given by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad \alpha \\ \min(X_{n-1}, \varepsilon_n) & \text{w.p.} \quad 1-\alpha \end{cases}.$$

where ε_n is a sequence of independent and identically distributed random variables independent of $\{X_n\}$. Then $\{X_n\}$ is stationary Markovian first order autoregressive Marshall-Olkin Weibull process with survival function $\bar{H}(x)$ if and only if ε_n has Weibull distribution with survival function $\bar{F}(x)$.

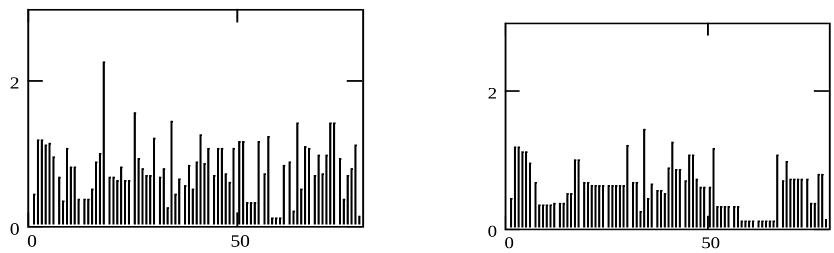
The proof and various properties can be established as above.

$$P(X_{n+1} > X_n) = \frac{\alpha}{2}$$

The joint survival function is

$$\begin{aligned} \bar{F}_{X_n, X_{n+1}}(x, y) &= P\left(X_n > x, X_{n+1} > y\right) \\ &= \left(\alpha \bar{F}_{X_n}(x) + (1-\alpha)\bar{F}_{X_n}(\max(x,y))\right)\bar{F}_{\varepsilon_n}(y) \\ &= \left(\alpha \frac{\alpha}{e^{\theta x^\beta} - \bar{\alpha}} + (1-\alpha)\frac{\alpha}{\max\left(e^{\theta x^\beta}, e^{\theta y^\beta}\right) - \bar{\alpha}}\right) e^{-\theta x^\beta} \end{aligned}$$

$$Corr(X_n, X_{n+1}) = 1-\alpha..$$



$\theta = 1 \quad \alpha = 0.8 \quad \beta = 2$ $\qquad$ $\theta = 1 \quad \alpha = 0.2 \quad \beta = 2$

Figure 5.2 Sample path behavior of the first order autoregressive Weibull process

Now we look in to the estimation of the first order autoregressive Weibull process. From the observed series, we find $P(X_{n+1} > X_n)$. If $P(X_{n+1} > X_n)$>.5 then we can conclude that the process is not a good fit. If $P(X_{n+1} > X_n) = .5$ then $\alpha = 1$ and we can see hat $Corr(X_n, X_{n+1}) = 1-\alpha = 0$ and $\{X_n\} \underline{\underline{d}} \{\varepsilon_n\}$.

If $P(X_{n+1} > X_n) < .5$ we can estimate the value of α by method of moments. Equate sample correlation to population correlation coefficient 1- α and find the value of α. Then the parameters θ and β can be estimated using the integral formula for $E(X^r)$ where X is a random variable following Marshall-Olkin Weibull distribution (see Marshall and Olkin(1997)).

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