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GENERALIZED $b^{\#}$- LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract
The aim of this paper is to introduce and study a new type of locally closed set called $gb^{\#}$-locally closed sets, which contain the class of $b^{\#}$-locally closed sets. Further the relations with other locally closed sets are investigated.

1. Introduction

The concept of locally closed sets was introduced by Bourbaki [4]. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Several mathematicians generalized this notion by replacing open sets with nearly open sets and generalized open sets and/or by replacing closed sets with nearly closed sets and generalized closed sets. In this paper, we introduce $gb^{\#}$-locally closed sets, $gb^{\#}$-lc^{*} sets, $gb^{\#}$-lc^{**} sets. The concept of locally closed sets has applications to separation axioms, functions and covering axioms.

2. Preliminaries

Throughout this paper (X, τ) or X represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space X, *cl*(A), *int*(A)denote the closure of A and the interior of A respectively.

We recall the following definitions, Corollary and Remarks which are useful in the sequel.

Definition 2.1

A subset A of a space X is called

(i) $b^{\#}$-open[10] if A=*cl*(*int*(A)) ∪ *int*(*cl*(A)) and $b^{\#}$-closed if $A = cl(int(A)) \cap int(cl(A))$.
(ii) semi-open [7] if A ⊆ *cl*(*int*(A)) and semi-closed if *int*(*cl*(A)) ⊆ A.(iii)pre-open [9] if A ⊆ *int*(*cl*(A)) and pre-closed if *cl*(*int*((A)) ⊆ A.
(iv) locally closed [5] if A = G ∩ F where G is open and F is closed.
(v) generalized b-closed set (briefly gb-closed) [1] if *bcl*(A) ⊆ U whenever A ⊆ U and Uis open.
(vi) generalized locally closed [2] if A = G ∩ F where G is g-open and F is g-closed.
(vii)generalized locally semi closed [3] if A = G ∩ F where G is g-open and F is semi-closed.

Definition 2.2 [11]

Let X be a space. A subset A of X is called generalized $b^{\#}$-closed (briefly $gb^{\#}$-closed) if $b^{\#}cl$(A) ⊆U whenever A ⊆ U and U is open. That is a subset A of a topological space X is $gb^{\#}$-closed if every open neighborhood of A contains its $b^{\#}$-closure.

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Proposition 2.3[12]
Every $b^{\#}$-open set is $gb^{\#}$ -open.

Definition 2.4 [10]
A subset A of a topological space X is called $b^{\#}$-locally closed if A = U ∩ F where U is $b^{\#}$-open and F is closed.

3. Generalized $b^{\#}$-locally closed sets

We introduce the following definition.

Definition 3.1
A subset S of X is called

(i) $gb^{\#}$-locally closed (briefly $gb^{\#}$-lc set) if S = A ∩ B where A is $gb^{\#}$-open and B is $gb^{\#}$-closed in X.

(ii) a $gb^{\#}$-lc^{*} set if there exist a $gb^{\#}$-open set A and a closed set B of X such that S = A ∩ B.

(iii) a $gb^{\#}$-lc^{**} set if there exist a open set A and a $gb^{\#}$-closed set B of X such that S = A ∩ B.

The collection of all $gb^{\#}$-locally closed sets is denoted by $Gb^{\#}$-LC(X, τ).

Remark 3.2
Every $gb^{\#}$-closed set is $gb^{\#}$-locally closed but the converse is not true as shown below.

Example 3.3
Let X = {a, b, c} and τ = {ϕ, X, {a}, {b}, {a, b}}. Then $Gb^{\#}$LC(X) = {ϕ, X, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. Then {a, b} is $gb^{\#}$-locally closed but not $gb^{\#}$-closed.

Definition 3.4
A space (X, τ) is called $T_{gb}{}^{\#}$-space if every closed set is $gb^{\#}$-closed.

Proposition 3.5
In a $T_{gb}{}^{\#}$-space, if A□$b^{\#}$-LC(X, τ) then A□$gb^{\#}$-LC(X, τ).

Proof
Let A□$b^{\#}$-LC(X, τ). Then by Definition 2.4, A = U ∩ F where U is $b^{\#}$-open and F is closed. Since every $b^{\#}$-open set is $gb^{\#}$-open by Proposition 2.3 and X is a $T_{gb}{}^{\#}$-space implies every closed set is $gb^{\#}$-closed. Then by using Definition 3.1, we have A□$gb^{\#}$-LC(X, τ).

Definition 3.6
A space (X, τ) is called $gb^{\#}$-space if every $gb^{\#}$-closed set is $b^{\#}$-closed.

Theorem 3.7
In a $gb^{\#}$-space every $gb^{\#}$-closed set is semi-closed and pre closed.

Proof

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Every $b^{\#}$-closed set is semi-closed and pre closed by Proposition 4.3 of [19]. Also by using Definition 3.6, X is a $gb^{\#}$-space that implies every $gb^{\#}$-closed set is $b^{\#}$-closed. This implies $gb^{\#}$-closed set is semi-closed and pre closed.

Definition 3.8

A space (X, τ) is called $gb^{\#}$-$T_{1/2}$ space if every $gb^{\#}$-closed set is closed.

Proposition 3.9

If (X, τ) be a $gb^{\#}$-space then $gb^{\#}\text{-}LC(X, \tau) \subseteq b^{\#}\text{-}LC(X, \tau)$.

Proof

Let x□$gb^{\#}\text{-}LC(X, \tau)$. Then by using Definition 3.1, $S = A \cap B$ where A is $gb^{\#}$-open and B is $gb^{\#}$-closed in X. Since X is a $gb^{\#}$-space, by using Definition 3.6, every $gb^{\#}$-closed set is $b^{\#}$-closed and X is a $gb^{\#}$-$T_{1/2}$ space implies by using Definition 3.8, every $gb^{\#}$-closed set is closed implies A is $b^{\#}$-open and B is closed in X. Then by Definition 2.4, x□$b^{\#}\text{-}LC(X, \tau)$. Therefore $gb^{\#}\text{-}LC(X, \tau) \subseteq b^{\#}\text{-}LC(X, \tau)$.

Proposition 3.10

If X is a $T_{gb}^{\#}$-space then $LC(X, \tau) \subseteq gb^{\#}\text{-}LC^{*}(X, \tau) \subseteq gb^{\#}\text{-}LC(X, \tau)$.

Proof

Let X be a $T_{gb}^{\#}$-space. Then by Definition 3.4, every open set is $gb^{\#}$-open and every closed set is $gb^{\#}$-closed.This implies $LC(X,\tau) \subseteq gb^{\#}\text{-}LC^{*}(X,\tau) \subseteq gb^{\#}\text{-}LC(X,\tau)$.

Proposition 3.11

Let (X, τ) be a $gb^{\#}$-$T_{1/2}$ space and $T_{gb}^{\#}$-space. Then

(i) $gb^{\#}\text{-}LC(X, \tau) = LC(X, \tau)$.

(ii) $gb^{\#}\text{-}LC(X, \tau) \subseteq GLC(X, \tau)$.

(iii) $gb^{\#}\text{-}LC(X, \tau) \subseteq GLSC(X, \tau)$.

(iv) $gb^{\#}\text{-}LC(X, \tau) = gb^{\#}\text{-}LC^{*}(X, \tau) = gb^{\#}\text{-}LC^{**}(X, \tau)$.

Proof

Let x□$gb^{\#}\text{-}LC(X, \tau)$. Then by using Definition 3.1, $S = A \cap B$ where A is $gb^{\#}$-open and B is $gb^{\#}$-closed in X. Since X is a $gb^{\#}$-$T_{1/2}$ space, by using Definition 3.8, every $gb^{\#}$-open set is open and $gb^{\#}$-closed set is closed implies x□$LC(X, \tau)$ by Definition 2.1(iv).
This implies $gb^{\#}\text{-}LC(X, \tau) \subseteq LC(X, \tau)$. Similarly we can prove $LC(X, \tau) \subseteq gb^{\#}\text{-}LC(X, \tau)$ by using $T_{gb}{}^{\#}$-space.This proves (i).
Also the proofs of (ii) and (iii) follow from (i), since for any space X, $LC(X, \tau) \subseteq GLC(X, \tau)$ and $LC(X, \tau) \subseteq GLSC(X, \tau)$ we have $gb^{\#}\text{-}LC(X, \tau) \subseteq GLC(X, \tau)$ and $gb^{\#}\text{-}LC(X, \tau) \subseteq GLSC(X, \tau)$. Let x□$gb^{\#}\text{-}LC(X, \tau)$. Then by using Definition 3.1, $S = A \cap B$ where A is $gb^{\#}$-open and B is $gb^{\#}$-closed in X. Since X is a $gb^{\#}$-$T_{1/2}$ space, by using Definition 3.8, every $gb^{\#}$-closed set is closed implies A is $gb^{\#}$-open and B is closed in X. Then x□$gb^{\#}\text{-}LC^{*}(X, \tau)$. Therefore $gb^{\#}\text{-}LC(X, \tau) \subseteq gb^{\#}\text{-}LC^{*}(X, \tau)$. Similarly every $gb^{\#}$-open set is open implies A is open and B is $gb^{\#}$-closed in X. Therefore $gb^{\#}\text{-}LC(X, \tau) \subseteq gb^{\#}\text{-}LC^{**}(X, \tau)$. Similarly the other inclusions can be proved by using $T_{gb}{}^{\#}$-space. Therefore $gb^{\#}\text{-}LC(X, \tau) = gb^{\#}\text{-}LC^{*}(X, \tau) = gb^{\#}\text{-}C^{**}(X, \tau)$. This proves (iv). □

Proposition 3.12

Let (X, τ) be a $gb^{\#}$-$T_{1/2}$ space and $T_{gb}^{\#}$-space. Then

(i) every lc-set is $gb^{\#}$-lc^{*} set.

(ii) every lc-set is $gb^{\#}$-lc^{**} set.

Proof

Let S be a locally closed set. Then S = A ∩ B where A is open and B is closed in X. Since X is a $T_{gb}^{\#}$- space, by Definition 3.4, every open set is $gb^{\#}$-open. This implies A is $gb^{\#}$-open and B is closed in X. Then by using Definition 3.1(ii), S is a $gb^{\#}$-lc^{*} set. This proves (i). Also X is a $gb^{\#}$-$T_{1/2}$ space implies every $gb^{\#}$-closed set is closed by Definition 3.8. This proves (ii).

□

Proposition 3.13

If X is a $T_{gb}^{\#}$-space then

(i) every $gb^{\#}$-lc^{*} set is $gb^{\#}$-lc set.

(ii) every $gb^{\#}$-lc^{**} set is $gb^{\#}$-lc set.

Proof

Let X be a $gb^{\#}$-lc^{*} set. Then by Definition 3.1(ii), there exist a $gb^{\#}$-open set A and a closed set B in X such that S = A ∩ B. Since X is a $T_{gb}^{\#}$-space, by Definition 3.4, every closed set is $gb^{\#}$-closed. Then by using Definition 3.1, X is a $gb^{\#}$-lc set. This proves (i).

Let X be a $gb^{\#}$-lc^{**} set. Then by Definition 3.1(ii), there exist a open set A and a $gb^{\#}$-closed set B in X such that S = A ∩ B. Since X is a $T_{gb}^{\#}$-space, again by using Definition 3.4, every open set is $gb^{\#}$-open. Then by using Definition 3.1, X is a $gb^{\#}$-lc set. This proves (ii).

□

Proposition 3.14

Let (X, τ) be a space with the condition that intersection of two $gb^{\#}$-closed sets is $gb^{\#}$-closed. Then the following are equivalent.

(i) A□$gb^{\#}$-LC(X, τ).

(ii) A = S ∩ $gb^{\#}$-*cl*(A) for some $gb^{\#}$-open set S.

(iii) $gb^{\#}$-*cl*(A) − A is $gb^{\#}$-closed.

(iv) A ∪ (X − $gb^{\#}$-*cl*(A)) is $gb^{\#}$-open.

(v) A ⊆ $gb^{\#}$-*int*(A ∪ (X − $gb^{\#}$-*cl*(A))).

Proof

Suppose (i) holds. Let A□$gb^{\#}$-LC(X, τ). Then there exist a $gb^{\#}$-open set S and a $gb^{\#}$-closed set G in (X, τ) such that A = S ∩ G. Since A ⊆ G and G is $gb^{\#}$-closed, $gb^{\#}$-*cl*(A) ⊆ G and S ∩ $gb^{\#}$-*cl*(A) ⊆ S ∩ G ⊆ A. Also A ⊆ S and A ⊆ $gb^{\#}$-*cl*(A) implies A ⊆ S ∩ $gb^{\#}$-*cl*(A) and therefore A = S ∩ $gb^{\#}$-*cl*(A). This proves (i) ⇒(ii).

Suppose (ii) holds. A = S ∩ $gb^{\#}$-*cl*(A) implies $gb^{\#}$-*cl*(A) – A = $gb^{\#}$-*cl*(A) ∩ (X−S) which is $gb^{\#}$-closed since X – S is $gb^{\#}$-closed and $gb^{\#}$-*cl*(A) is $gb^{\#}$-closed. This proves (ii) ⇒ (iii).

Suppose (iii) holds. A ∪ (X − $gb^{\#}$-*cl*(A)) = (X − ($gb^{\#}$-*cl*(A) − A)) and by assumption, (X − ($gb^{\#}$-*cl*(A) − A)) is $gb^{\#}$-open. This implies A ∪ (X − $gb^{\#}$-*cl*(A)) is $gb^{\#}$-open. This proves (iii) ⇒ (iv).

Suppose (iv) holds. By assumption, A ∪ (X − $gb^{\#}$-*cl*(A)) = $gb^{\#}$-*int*(A ∪ (X − $gb^{\#}$-*cl*(A))). This implies A ⊆ $gb^{\#}$-*int*(A ∪ (X − $gb^{\#}$-*cl*(A))). This proves (iv) ⇒ (v).

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Suppose (v) holds. By assumption and since A ⊆ gb$^{\#}$-*cl*(A), A = gb$^{\#}$-*int*(A ∪ (X − gb$^{\#}$-*cl*(A))) ∩ gb$^{\#}$-*cl*(A). Therefore A□gb$^{\#}$-LC(X, τ). This proves (v) ⇒ (i).

□

Theorem 3.15

Let (X, τ) be a space with the condition that intersection of two gb$^{\#}$-closed sets is gb$^{\#}$-closed. Then the following are equivalent.

(i) A□gb$^{\#}$-LC*(X, τ).

(ii) A = S ∩ *cl*(A) for some gb$^{\#}$-open set S.

(iii) *cl*(A) − A is gb$^{\#}$-closed.

(iv) A ∪ (X − *cl*(A)) is gb$^{\#}$-open.

Proof

Let A□gb$^{\#}$-LC*(X, τ). There exist an gb$^{\#}$-open set S and a closed set G such that A = S ∩ G. Since A ⊆ S and A ⊆ *cl*(A), A ⊆ S ∩ *cl*(A). Also since *cl*(A) ⊆ G, S ∩ *cl*(A) ⊆ S ∩ G = A. Therefore A = S ∩ *cl*(A). This proves (i) ⇒ (ii).

Since S is gb$^{\#}$-open and *cl*(A) is a closed set, A = S ∩ *cl*(A)□gb$^{\#}$-LC*(X, τ). This proves (ii) ⇒ (i).

Since *cl*(A) − A = *cl*(A) ∩ (X − S) and also since intersection of a gb$^{\#}$-closed set and a closed set is gb$^{\#}$-closed, *cl*(A) − A is gb$^{\#}$-closed. This proves (ii) ⇒ (iii).

Let S = X − (*cl*(A) − A). Then by assumption S is gb$^{\#}$-open in (X, τ) and A = S ∩ *cl*(A). This proves (iii) ⇒ (ii).

Let G = *cl*(A) − A. Then X − G = X − A ∪ (X − *cl*(A))and A ∪ (X − *cl*(A)) is gb$^{\#}$-open. This proves (iii) ⇒ (iv).

Let S = A ∪ (X − *cl*(A)). Then X – S is gb$^{\#}$-closed and X − S = *cl*(A) − A and so *cl*(A) − A is gb$^{\#}$-closed. This proves (iv) ⇒ (iii).

Definition 3.16

A space is said to have the gb$^{\#}$-closure preserving property if gb$^{\#}$-*cl*(A) is always gb$^{\#}$-closed.

Theorem 3.17

Suppose X has the gb$^{\#}$-closure preserving property and let A be a subset of (X, τ). Then A□ gb$^{\#}$-LC**(X, τ) if and only if A = S ∩ gb$^{\#}$-*cl*(A) for some open set S.

Proof

Let A □ gb$^{\#}$-LC**(X, τ). Then A = S ∩ G where S is open and G is gb$^{\#}$-closed. Since A ⊆ G and A ⊆ gb$^{\#}$-*cl*(A), A ⊆ S ∩ gb$^{\#}$-*cl*(A). By Definition 3.16, gb$^{\#}$-closure of A is gb$^{\#}$-closed. Then gb$^{\#}$-*cl*(A) ⊆ G and hence S ∩ gb$^{\#}$-*cl*(A) ⊆ S ∩ G = A. Therefore A = S ∩ gb$^{\#}$-*cl*(A) for some open set S. Conversely assume that A = S ∩ gb$^{\#}$-*cl*(A) for some open set S. By Definition 3.16, gb$^{\#}$-closure of A is gb$^{\#}$-closed in X. Therefore A□gb$^{\#}$-LC**(X, τ).

Corollary 3.18

Let A be a subset of (X, τ). If A□gb$^{\#}$-LC**(X, τ), then gb$^{\#}$-*cl*(A) − A is gb$^{\#}$-closed and A ∪ (X − gb$^{\#}$-*cl*(A)) is gb$^{\#}$-open.

Proof

Let A□gb$^{\#}$-LC**(X, τ). Then by Theorem 3.15, A = S ∩ gb$^{\#}$-*cl*(A) for some open set S and gb$^{\#}$-*cl*(A) – A = gb$^{\#}$-*cl*(A) ∩ (X − S) is gb$^{\#}$-closed in (X, τ). If G = gb$^{\#}$-*cl*(A) − A, then X − G = A ∪ (X − gb$^{\#}$-*cl*(A))and X – G is gb$^{\#}$-open. This implies A ∪ (X − gb$^{\#}$-*cl*(A)) is gb$^{\#}$-open.

CONCLUSION

Thus we studied gb$^{\#}$-locally sets, gb$^{\#}$-lc** sets and gb$^{\#}$-lc* sets and their properties are investigated.

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